Mathematics Inside the Black Box: Assessment for Learning in the Mathematics Classroom

Jeremy Hodgen and Dylan Wiliam

1. Aims of mathematics teaching

Claims of widespread innumeracy in the general population abound, while employers and academics alike complain that in mathematics, school pupils are ill-prepared for employment or further study. As a result, the current focus of attention is towards mathematical literacy for all, through reform of the 14-19 qualifications and examination system, and the introduction of 'functional mathematics' qualifications (DfES 2005; Smith 2004).

While curricular changes may affect what is taught, the fundamental issues and problems of mathematics education remain. The power of mathematics lies in its universality and the framework it provides with which to interact with the world. Mathematics is a *connected* body of knowledge. To be successful pupils need to build up what Skemp (1976) calls a *relational* understanding of how ideas interrelate. Yet pupils at all levels of achievement have difficulty transferring and connecting the mathematics of the classroom into other contexts. Many view mathematics simply as an esoteric set of arbitrary procedures useful only in mathematics classrooms and examinations. Mathematical literacy requires understanding of the meaning, use and justifications of mathematical ideas.

Formative assessment fits well with these aims, since its purpose is to help teachers to filter the rich data that arise in class discussion and activity, so that professional judgments can be made about the next steps in learning. At specific times, learners also have to prepare for examinations. While there needs to be a focus on mathematical literacy to support this, there also has to be time set aside near the end of courses to hone examination techniques. Feedback, peer and self-assessment have important roles to play in this process and, utilized properly, formative assessment can result in large learning gains.

2. Principals of learning

Three types of feedback are essential to formative assessment. The first is from student to teacher; the second is from teacher to student; and the third is between students. Learning is effected by judicious use of all these, in which each contribution responds to the other. Thus, for classroom dialogue, the starting point is generally a question formulated by the teacher to put 'on the table' the ideas that are the starting point for the students. This implements the *first* principle of learning, which is to *start from where the learner is*, recognizing that students have to reconstruct their ideas and that to merely add to those ideas an overlay of new ideas tends to lead to an understanding of mathematics as disconnected and inconsistent.

But that is not enough: the teacher then has to encourage, and to listen carefully to, a range of responses, taking them all seriously, whether they be right or wrong, to the point, or not, and helping students to talk through inconsistencies and respond to challenges. In such discussion the teachers are fashioning their interventions to meet the learning needs that have been made evident, but they are also implementing a *second* principle of learning, which is that *students must be active* in the process – learning has to be done *by* them; it cannot be done *for* them.

The *third* principle is that *students need to talk about their ideas*. When pupils are talking about mathematical ideas, whether in a whole-class dialogue or in peer groups, they are using and constructing the language of mathematics. 'Talking the talk' is an important part of learning. Teachers are rightly critical of professional development that does not allow them the space to express their ideas and to explore ways in which new inputs might make sense to them and to their colleagues. Their pupils need the same opportunities.

A *fourth* principle is that in order to learn, *students must understand the learning intention*, which requires understanding of what would count as a good quality work (success criteria). They must also have an idea of where they stand in relation to that target. Only with these two can they achieve the power to oversee and steer their own learning in the right direction, so that they can take responsibility for it (this is part of what psychologists call metacognition). This is no easy undertaking, and it requires attention in teaching to helping students understand the targets of the learning work and the criteria of quality, i.e. to be able to tell whether a product of their efforts does or does not meet the criteria. However, simply providing lists of criteria of what makes for a good piece of mathematics is rarely sufficient to help pupils' progress. Rather, pupils need to engage in mathematical argument and reasoning in order that they and their peers can learn the ways in which the quality of mathematical work is judged. Peer and self-assessment are essential to this process, for they promote both active involvement and practice in making judgments about the quality of work – their own and of fellow students.

The *fifth* principle is that *feedback should tell pupils how to improve*. When feedback focuses on the student as a good or bad achiever, emphasizing overall judgment by marks, grades or rank order lists, it focuses attention on the self (what researchers have called ego-involvement). A synthesis of 131 rigorous scientific studies showed that this kind of feedback actually lowered performance (Kluger and DeNisi 1996). In other words, performance would have been higher if no feedback had been given. This is because such feedback discourages the low attainers, but also makes high attainers void tasks if they cannot see their way to success, for failure would be seen as bad news about themselves rather than as an opportunity to learn. In contrast, when feedback focuses not on the person but on the strengths and weaknesses of the particular piece of work (task-involving feedback) and what needs to be done to improve, performance is enhanced, especially when feedback focuses not only on what is to be done but also on how to go about it. Such feedback encourages all students, whatever their past achievements, that they can do better by trying, and that they can learn from mistakes and failures (see Dweck 1999).

These principles make substantial demands on teachers' subject knowledge, not only to make sense of what pupils say but also to be able to determine what would be the most appropriate next steps for the pupil. This is not the abstract knowledge gained from advanced study in mathematics, but rather a 'profound understanding of fundamental mathematics' (Ma 1999).

3. Classroom dialogue: talking in and about mathematics

Talking is central to our view of teaching mathematics formatively. One of the strengths of mathematics lies in the way that ideas and concepts can be expressed in a very concise form. Yet this strength makes mathematics difficult to teach and learn. Hence, providing opportunities for students to express, discuss and argue about ideas is particularly important in mathematics (Hodgen and Marshall 2005). Through exploring and 'unpacking' mathematics, students can begin to see for themselves what they know and how well they know it. By listening to and interacting with pupils, a teacher can provide feedback that suggests ways in which pupils can improve their learning. Feedback, whether from teacher or pupils, is useful to both in providing information that enables both to modify the teaching and learning activities in which they are engaged.

Implementing such an approach is a complex activity that involves the following aspects:

- challenging activities that promote thinking and discussion;
- encouraging pupil talk through questioning and listening;
- strategies to support all learners to engage in discussion;
- peer discussion between students; and
- rich and open whole-class discussions.

Of course, these issues are about much more than formative assessment – they touch on all aspects of teaching and learning. In this section we examine how these issues can contribute to generating and acting upon formative opportunities.

Challenging activities

If mathematics teachers want to find out what children understand in mathematics, rather than just what they can recite, then their pupils need to be challenged by activities that encourage them to think and talk about their ideas. This may involve presenting students with the unexpected: an 'obvious' answer that is in some way inadequate, a problem that does not have one correct answer or a teacher defending the 'wrong' answer. Often a problem with formative potential is one in which, paradoxically, pupils who know more are more likely to get it 'wrong', and in doing so reveal to themselves or the teacher something about the way in which they understand the mathematics in question. So, in addition to providing opportunities for students to construct mathematical ideas, challenging activities generally also constrain pupils' thinking. But working formatively is not simply about finding out what pupils know currently. Hence, these problems also provide opportunities for feedback on what to do next.

The obvious answer is not always correct

Much of school mathematics consists of exercises in which answers are either right or wrong. Posing problems in which the most obvious answer is either wrong or only partially correct encourages pupils to defend their ideas. The ensuing discussion can provide an opportunity for learners to examine their ideas and how well they know them. In the following activity students are presented with a multiple-choice problem:

Which one of these statements is true?

A. 0.33 is bigger than 1/3
B. 0.33 is smaller than 1/3
C. 0.33 is equal to 1/3
D. You need more information to be sure

Some students may think that they are the same and opt for C; other may opt for B since 0.33 is smaller than 0.333; while D is a more general answer, because A could be correct (0.33 rounded to 2 dp could be as large as 0.33499999). All these answers are to some degree 'correct' and 'justifiable'. The problem provides an opportunity to differentiate between different levels of understanding of place value and of the equivalence of decimal and vulgar fractions. The extent to which this activity works formatively is dependent on the extent to which the teacher succeeds in encouraging dialogue and discussion and on the quality of his or her interactions. In order to probe understanding she might ask 'Could 0.33 ever be bigger than 1/3? 'Is .033 always equal to 1/3?'.

The next steps in learning are often indicated in students' responses – either a student's own responses or those of others. The teacher has a crucial role to play in finding ways to make these next steps explicit. Doing so may involve questions which ask pupils to extend their knowledge into new areas such as: 'Is this the same for 0.25 and ¼?'. Alternatively, the teacher might ask students to reflect on the differences between decimal and vulgar fraction notation: 'When would you use a decimal fraction and when would you use a vulgar fraction?'.

There are two particularly important features of this activity that facilitate formative assessment:

- The challenges are directed at a wide range of ability and achievement levels providing an opportunity for pupils to learn from each other.
- Increasing ability and achievement does not necessarily increase a pupils' likelihood of getting a 'correct' answer. Correct answers generally offer fewer opportunities both to assess how well pupils grasp a mathematical concept and to offer feedback on how they could improve or adapt their ideas.

Using what we know about pupils' mathematical understanding

The mathematics curriculum is content-heavy. There is a lot to learn and limited time to which to learn it. As a result, even successful pupils can have difficulties with relatively simple ideas in new or unusual contexts. For example, faced with question $\sqrt{0.4} = ?$ [$\sqrt{10/5}$, 0.63], many students, even high achievers, give an incorrect answer of 0.2. Crucially, here, the context is one that lulls the student into giving an incorrect response. The digit, 4, instantly recognizable to a good mathematician as 2 squared, is designed to distract the student's attention. Of course, high-achieving students can readily calculate 0.2 squared and notice an error. Asking a question (e.g. 'What is confusing about that problem?') can focus students' attention on what they need to do to improve their understanding. There is a great deal of published research on the way students understand mathematics on which teachers can draw.

Problems with more (or less) than one correct answer

Students generally expect mathematical problems to have one and only one correct answer. Yet 'real' mathematical problems may have many solutions or none:

- Some problems have a range of mathematically correct solutions, e.g. the quadratic equation $x^2 = 4$ (+2, -2).
- Open-ended problems may enable pupils to specify some of the criteria for themselves, e.g. 'In what ways could this sequence be continued: 1, 2, 4 . . . ' (e.g. 7, 11 . . . or 8, 16 . . ..).
- Modeling problems may require interpretation in order to assess how well they fit the real world, e.g. 'What does this statistical evidence about mobile phone use tell us about whether mobile phones are safe?'.
- Some problems have no solution, e.g. draw a triangle with sides 4cm, 6cm and 11 cm.

By challenging pupils' expectations of mathematics, these problems can provide an opportunity to provoke discussion and disagreement among pupils. When asked what fraction the following diagram represents, pupils may quickly say 4/10 or 2/5. The diagram could equally well represent 3/5, 6/10, 2/3, 3/2 or 6/4:

Challenging students by asking them what fractions *could* be represented by the diagram provides an opportunity to explore and assess their understandings of part-whole and part-part relationships. The open-endedness of the problem provides an opportunity to suggest ways in which pupils could extend their understandings by asking them to find other diagrams or images to represent the different fractions.

Problems with no solution can be equally productive. The following simultaneous equations have no solution:

$$5 = 6y + 2x$$
$$2 = x + 3y$$

Generally, students learn to solve simultaneous equations using algebraic methods: by substitution or by elimination. These are good methods which all students should aim to be able to do well. In this case, these both result in the incorrect 'equality' 5 = 4 (or something similar) since the equations represent parallel lines. Typically, students are able to carry out the algebraic manipulation and the graphical methods necessary to solve this problem but see little connection between the two different forms of representation. Hence, students think they must have made a simple algebraic error, and check and recheck their working. The problem provides an opportunity for the teacher and the students themselves to assess what the students know: how well they can carry out the algebraic manipulation or use graphical methods (their procedural knowledge) and the extent to which they understand what lies behind this (their conceptual knowledge). It also provides the context to give feedback on what students can do to improve their learning.

Probing the students' knowledge might involve the teacher playing devil's advocate: 'Five doesn't equal four. Your algebra must be wrong'.

Used sensitively, giving students the opportunity to prove the teacher 'wrong' can be a powerful way of promoting independent learning. Alternatively, the teacher might ask a pupil to explain her thinking: 'Paula, you say there's no solution. How do you k now?'.

Feedback on ways of improving conceptual understanding might come in the form of teacher questions such as: 'Can you represent these equations in a different way? Could you use a graph?'.

Alternatively, feedback may come from the pupils themselves in listening to, adding to and improving the ideas of others.

Generating mathematical structure
Identifying similarities and differences can enable pupils to begin to generate mathematical structures for themselves. For example, to experienced mathematicians and mathematics teachers, all quadratic equations are essentially similar. This structure is much less clear to pupils encountering them in school mathematics lessons. One teacher, for example, gave students a set of 20 quadratic equations, each expressed algebraically and written on a card. She asked pupils in pairs to sort the equations into groups of no more than five. In the discussion that followed, the teacher's questions included:

Sara, what do you think is similar about $y = 3x^2 + 4$ and $y = 3x^2 - 1$? What would the graphs of these equations look like?

Mike, you put those two equations in different groups. What do you think is different about the two equations?

What would the graph of $y = 3x^2 = 21$ look like?

Two crucial factors in this activity's strength were that the teacher had structured it so that different groups of pupils grouped the equations in different ways, thus providing some mathematical disagreement, and each pair had considered *all* the equations thus enabling *all* pupils to engage in the discussion.

'Closed' questions can sometimes be valuable
Closed questions have come in for much criticism in mathematics education, but some closed questions can be very powerful. The teacher could ask the following questions (using say, mini-whiteboards to get a response from all the class): 'Are all squares rectangles?'. If all students get the answer correct, the teacher can move on. If no-one gets it correct then the teacher might re-teach the definitions. But if part of the class get it right and part get it wrong, the teacher can organize a discussion.

You thought the statement was true. Why?

You thought the statement was false. Why?

The use of provocative statements like '1/10 is double 1/5' are particularly powerful in mathematics and often more powerful than a more direct question (Dillon 1990). This can provide an opportunity for pupils to challenge the teacher and for pupils to debate a particularly difficult aspect of fractions.

Generating different solutions
Often we give students the message that school mathematics is about getting answers to problems, whereas our actual aim is to enable them to learn mathematics. Asking pupils to generate different ways of solving a problem is one way of focusing their attention on the process of mathematics. The following activity asks pupils to find different ways of solving what is essentially a quadratic equation:

If $a + b - 2$, how big could ab be? Find different methods of solving this problem using calculus, algebra and co-ordinate geometry.

Through examining and comparing different techniques, the pupils can assess their own mathematical strengths. Knowing one solution can help pupils generate and understand another, and this can enable them to understand the connections between different mathematical domains. Feedback might be in the form of a reflection on the activity:

What are the advantages and disadvantages of the methods?

What is similar . . . what is different about the ways of solving the problem?

Did you find one method easier than another?

Encouraging pupils to unpack and share the ideas they consider easy and hard can provide them with some insight into strategies that they find difficult.

Mistakes are often better for learning than 'correct' answers

Activities that focus on identifying and correcting common errors can be helpful in both the assessment and the feedback stages of formative assessment. For example, in the following division calculation, there is a place value error in setting out the first stage of the subtraction:

```
      21r9
21 | 472
     42
      30
      21
       9
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A faulty calculation for 472 ÷ 21 [Correct answer: 22 r 10]

This faulty calculation could be presented among a set of similar calculations – some with errors, others without – and pupils asked to identify which are correct or incorrect. Asking pupils to find what has gone wrong in the use of this algorithm can help some pupils to identify the mistakes they make. More importantly, by focusing on the process of the division calculation rather than its result, pupils can identify why such errors are made. This in turn can help pupils understand what they know well and what they know less well. Feedback could take the form of pupils providing advice to others on why such errors happen and how to avoid them.

Mistakes and errors arise naturally in learning. Pupils' errors are invaluable for teaching and learning provided the classroom is one in which these mistakes are valued. Many of the activities that we describe place pupils in situations in which they encounter unexpected and unusual results – results that are mathematically correct but which the pupils think are wrong; or results that are mathematically incorrect but which the pupils think are right. Piaget called this cognitive conflict and argued that through resolving conflict students can make leaps forward in their understanding.

Using textbooks

Mathematics textbooks vary in quality. Some consist largely of predictable exercises which practice procedures, but all textbooks can be used as a starting point for formative teaching. For example, pupils could be asked to identify four questions, two which they consider easy and two which they consider difficult. They could then construct model answers – working individually on the 'easy' questions and with a partner on the 'difficult' ones. Pupils might then be asked:

What is similar . . . what is different about the easy and hard questions?

Have you changed your views on which are easy and which are hard questions?

How could you make that question easier/harder?

What advice would you give on how to solve a hard problem?

Where pupils' judgments are different, different groups of pupils can be asked to present a justification and explanation of their view. Alternatively, pupils could ask for advice from the class on how to solve a 'difficult' question.

At the end of a lesson sequence, pupils could be asked to produce an alternative to the textbook with explanations and problems, providing advice and guidance to others. Pupils could work, first, individually or in pairs, and then in larger groups. This would provide an opportunity for pupils first to find out what they know, then to compare this with the ideas of others.

Using summative tests formatively

Considerable money and expertise has been used to develop the many summative tests that students take in schools in England. Teachers in the KMOFA Project used external tests and tests that they had constructed themselves using items from external tests as tools to be used formatively in the classroom. Some teachers stopped the habit of explaining the mark scheme when tests were returned, and instead analyzed the tests to see which specific questions were causing most problems for the class. They then used the time after marking to rework the ideas behind the difficult questions and to give further examples of these for students to try. For other test questions, where only a few students had answered incorrectly, students were told to find someone in the class who had answered correctly and get them to explain how they arrived at their answer. The teacher dealt with serious gaps in understanding but smaller gaps could be closed through peer activity. Other ways of using summative tests include:

- Give the students the mark scheme for the test and ask them to construct model 'full mark' answers.
- Ask pupils to identify easy and hard questions. Pupils can then test their hunch by explaining the 'easy' questions to others – and by asking other pupils to explain the more difficult questions.
- Ask pupils to answer a test individually and hand it in, then to work with a partner on the questions they found difficult, and finally, attempt to improve their original answers individually. Alternatively, pupils can answer a test individually, but then be asked to work in groups to produce the best composite answers they can. The teacher can then lead a plenary discussion by asking each group for their best answers to each question.
- Use a test halfway through a sequence of lessons to identify the areas pupils do not understand fully.
- Give pupils a test and ask them, in pairs, to produce a more difficult test. Pupils would have to produce solutions for the questions as well as justify in what ways their test was more difficult.

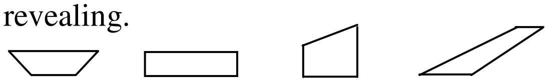
What is common to all of these activities is that summative tests are used to enable pupils to probe their own understandings and to get feedback for themselves, from other pupils and from the teacher on ways in which to improve.

Good problems are not universal

None of these activities will work with all children at all times. The extent to which pupils can respond to any of these challenges depends very much on their existing knowledge; to be challenged here is dependent on knowing some relevant mathematics. The realization of the formative potential within these activities has to be facilitated by the teacher through finding out what students already know and challenging them to extend their knowledge. Of course, posing a problem and listening to the pupils' ideas may demonstrate that pupils have no difficulty in the area, in which case the teacher would move on. On the other hand, it may indicate that a topic needs talking again, perhaps in a different way.

Generating challenging activities

The examples we discuss in this pamphlet are small in number and can only be illustrative of the approach. Most importantly, there is no such thing as a universally challenging activity. An activity that works well with one group of students may fall completely flat with a different group simply because it fails to engage them on that occasion. Asking pupils which of the following shapes is a trapezium is pointless before they have learned the definition of a trapezium, and probably not very useful when the pupils have a thorough understanding of trapeziums. But there is a point at which the activity is useful, thought-provoking and revealing.



The task of generating appropriate and challenging activities can only be done by teachers themselves. In teaching, developing 'new' activities usually involves borrowing and adapting old ideas. Most of the starting points that we present here are 'old chestnuts' that will be familiar to many readers. To come up with these ideas, teachers need to do what the KMOFA Projects teachers did in order to construct formative activities for themselves. Starting from existing activities, and *working with other teachers*, the project teachers looked for opportunities to put the principals of learning outlined above into practice. While doing this, the following questions may be helpful.

- In what ways does this activity promote mathematical learning and talk?
- What opportunities are there for the teacher and pupils to gain insights into the pupils' learning?
- How does this activity enable the teacher and the pupils to understand what the pupils need to do next?

In other words, the key questions is, 'What is formative about this practice?'.

Answering these questions is very much easier after a teacher has used an activity several times, because he or she can anticipate some of the ways in which different pupils will react to it. Comparative research with Japanese teachers suggests that one very successful strategy is for teachers to work collaboratively to hone and perfect a *small* number of activities (Stigler and Hiebert 1999). By doing this – and sharing their ideas with others – these teachers increased their repertoire of activities and improved their understanding of the curriculum as a whole.

There are many resources from which teachers can draw useful starting points. These include teaching programmes and books on thinking mathematically (e.g. Adhami *et al.* 1998; Mason *et al.* 1985), advice on developing and adapting activities (e.g. Prestage and Perks 2001), assessment materials (Brown 1992), websites (e.g. NRICH – see References) and materials published by the Association of Teachers of Mathematics and the Mathematical Association (see References).

Encouraging pupil talk through questioning and listening

In mathematics classrooms, teachers tend to work too hard while the pupils are not working hard enough, resulting in the old joke that schools are places where children go to watch teachers work. Where students are actively involved in discussion, not only do they learn more but also their general ability actually increases (Mercer *et al.* 2004). This is only possible, however, if classroom discussion develops beyond a series of rapid-fire, closed questions, which often only include a few pupils and allow little time for reflection, towards an atmosphere where the activities are so structured that they offer real opportunities for thinking. Of course, teacher interventions are crucial in promoting formative assessment. But we are suggesting that these interventions should, in general, be less frequent but more thoughtful and challenging. The ensuing classroom culture has a number of benefits:

- By listening more to pupils, teachers learn more about what pupils know and how well they know it.
- More pupils have more opportunity to express their ideas through longer contributions. They have more opportunity to listen to and compare their own ideas to those of others, and thus more opportunities to learn from their peers.

- By being listened to, pupils realize the teacher is actually interested in what they say and are thus encouraged to say more.
- Talking less gives the teacher more time to think about the interventions he or she *does* make.

There are two broad aspects to teachers' questioning: initiating formative activity and responding to pupils. We have discussed various approaches to initiating formative assessment activity in the section on challenging activities above. When planning activities, teachers do, of course, need to anticipate how pupils may respond and to generate appropriate interventions and questions. The topic of generating powerful questions is one that deserves a book in itself and is tackled well by others (e.g. Watson and Mason 1998). We confine ourselves here to highlighting a few generic question types:

Tell me about the problem. What do you know about the problem? Can you describe the problem to someone else?

Have you seen a problem like this before? What mathematics do you think you will use?

What is similar . . . ? What is different . . . ?

Do you have a hunch? . . . a conjecture?

What would happen if . . . ? Is it always true that . . . ? Have you found all the solutions?

How do you know that . . . ? Can you justify . . . ? Can you prove that . . . ?

Can you find a different method?

Can you explain . . . improve/add to that explanation . . . ?

What have you found out? What advice would you give to someone else about . . . ?

What was easy/difficult about this problem . . . this mathematics?

Of course, questioning is more complex than simply generating questions. Responsive questioning – responding in the moment to pupils' ideas – is very complex. There are no easy answers to this, but teachers in the KMOFA Project found collaboration – sharing, talking about and reflecting upon questioning with other teachers – to be a very valuable way of increasing their repertoire of questions *and* their ability to use these questions in the classroom.

One of the most important changes that we observed in the practice of the KMOFA Project teachers was in the way they listened to pupils' responses. At the beginning of the study, teachers listened to pupils' responses in a way that Brent Davis calls 'evaluative listening' (Davis 1997). They listened for, the correct answer, and when pupils gave partially correct answers they said things like 'Almost' or 'Nearly'. This encouraged the belief that the teachers were more interested in getting pupils to give the correct answer, rather than finding out what they thought about a particular idea. Over time, the teachers increasingly listened *interpretively* to the pupils – they listened to what pupils said in order to work out why the pupils had responded in the way they had. What was more, even the pupils noticed. As one Year 8 girl said, 'When Miss used to ask a question, she used to be interested in the right answer. Now she's interested in what we think'.

Strategies to support all learners

Demanding tasks require time and space for learners both to engage with the challenge and to generate responses. One strategy for higher-order questions is to increase the wait time (the time between a teacher asking a question and taking an answer). The wait time in many mathematics classrooms is very low, less than one second. The fairly simple strategy of increasing wait time to around three seconds can have very dramatic effects on the involvement of students in classroom discussion (Askew and Wiliam 1995). This includes:

- longer answers;
- more students contributing;
- more students commenting on or adding to the contributions of others; and
- a greater range of explanations and examples offered.

While increased wait time is very powerful, it is not a universal panacea. Increased wait time is not an effective strategy for lower-order, more straightforward questions (e.g. recall of number facts), and solely increasing wait time to more than about five seconds can actually decrease the quality of classroom talk (Tobin 1986).This suggests that pupils need structure as well as time. Our own research suggests a number of other related strategies:

- Encourage pupils to jot down an answer on either piece of paper or a mini-whiteboard, so that when asked by the teacher to answer they can use their jotting as an *aide-memoire.*
- Give pupils a set time in which to discuss the problem with a partner and generate a contribution. The actual time would depend on the question but might vary between 30 seconds and two minutes.
- Teachers can 'rehearse' pupils' contributions with them prior to a class discussion.
- A 'no hands up' strategy can help avoid the same high-achieving pupils making the majority of contributions and establish a classroom culture in which all pupils are expected to have a valuable contribution to make. If a pupil responds 'I don't know', an effective strategy can be to say 'OK, I'll come back to you'. After the teacher has collected answers from other students, he or she can return to the first student and ask which of the answers is best. In this way, the pupil needs to listen to other pupils and engage in the activity, rather than just brood about failure.
- Give pupils transparent page protectors and dry-wipe pens to record their ideas. A graph grid can be inserted and used by the pupils to plot a graph. By taking the opaque insert out, students (or the teacher) can display and present their work to the whole class.
- When pupils make a higher-order contribution – whether as a new idea or a question – give the other pupils wait time before taking responses. Or ask pupils to discuss the idea or question in pairs.
- Begin a class discussion by taking one idea or comment from each pair or group, recording each idea on the board.
- Encourage pupils to make hunches or conjectures even if they can't explain them. Other pupils may be able to help them explain.
- Invite pupils to say when they don't understand even if they cannot frame a suitable question. Talking can help them or prompt another pupil.
- Give pupils time to think about and disagree with an idea *even if it is correct,* e.g. 'Luke says that ¾ is the same as 12/16. Do you agree . . . does that mean the two fractions are exactly the same?'.

Questions and statements can be also used to encourage contributions:

Can you put Amy's idea into your own words?

What can we add to Saheera's answer?

Well, if you're confused you need to ask Jack a question.

Which parts of Suzie's answer would you agree with?

Can someone improve on Simon's answer?

Opportunity for peer discussion

Peer discussion plays an essential part in the formative classroom. Discussion in small groups enables all pupils to engage directly in discussion about the mathematical problem. By doing so, they are better able to understand the problem and they can clarify their own ideas. As a result, a greater number of pupils contribute to whole-class discussions and their contributions are better articulated. Our research suggests that more frequent, but shorter, whole-class discussions balanced with small-group discussions are more effective in encouraging focused peer discussion about mathematics. Other strategies include:

- Ask pupils to discuss an idea in pairs. When sharing their ideas, toss a coin to decide which partner will report on their discussions

to the class. This encourages all pupils to be ready to contribute. Alternatively, asking pupils to report on their partner's ideas encourages them to listen to each other.
- Use the jigsaw technique. Ask pupils to work in groups of 4-6 on different, but related, problems, then each member of the group joins a different mixed group to share what they have done. This enables each pupil to be an 'expert' on part of a problem.

Encouraging open discussion

Our research suggests that discussion in which all pupils contribute openly is vital to effective, formative feedback. Students can at times be reluctant to give answers that they think may be incorrect. Hence, teachers need to value all mathematical contributions – mistakes and partially correct answers included – and encourage pupils to challenge any ideas they disagree with or do not understand.
- Record all contributions on the board. Ask students, in pairs, to identify one idea they disagree with or do not understand. Students can then ask for clarification or explanation in whole-class discussion.
- Challenge correct as well as incorrect answers. Ask pupils to justify their ideas: 'How do you know . . . ?' and ask other pupils to comment: 'What do you think about . . . ?'. Thus pupils are asked to think about *how* as well as *what* they know.
- Practice ways of avoiding letting pupils know whether an answer is correct through body language, facial expression or type of comment.

One useful technique is for teachers to make a mistake deliberately. If pupils spot the mistake, this provides an opportunity for them to correct the teacher and provide an explanation in their own words. If the pupils don't notice it, the teacher could highlight it him or herself: 'Oh, that doesn't seem quite right, can anyone spot what I've done wrong?'. Giving students responsibility for identifying and correcting errors in this way encourages an attitude that what matters are the arguments that support mathematical ides rather than whether the person saying them is 'good' at math or not.

4. Feedback and marking

Before working on formative assessment, most of the teachers we worked with marked students' work with ticks and crosses, summing the number of ticks to produce an overall 'mark'. Although this is a quick method of marking, it is of very little use as formative feedback for several reasons:
- marks do not give learners the advice on how the work – or the learner's understanding – can be improved;
- marks emphasis competition not personal improvement; and
- marks demotivate low attainers and provide no challenge to high attainers.

Teachers often compromise by using marks and comments. However, research (Butler 1988) indicates that this is a waste of time on the teacher's part because the students only focus on the mark and fail to read the comment which would provide the advice for improvement. But comment-only marking can only be effective where it helps learners know where they are and where they should go next, by offering specific advice and targets.

Comments are the most common way for the teacher to have a dialogue with each individual pupil. While comments take a long time to write, the KMOFA Project teachers found it worth doing. Useful comments written every two to three weeks were more useful than a mark on every piece of work. Thus, instead of marking each piece of work, the teacher can:
- provide specific feedback on a particular aspect of a pupil's work;
- identify particular patterns of errors in a pupil's work;
- give structured feedback that enables a pupil to identify errors for themselves; and
- encourage pupils to use their existing knowledge in assessing their own work.

The content of effective written comments, of course, varies according to the activity and mathematical content. Often, advice to pupils will be very similar to the kinds of interventions and questions that a teacher uses with the whole class, although it is an opportunity for the teacher to give personalized feedback. Specific strategies include:

Enabling pupils to identify the errors for themselves:

- There are five answers here that are incorrect. Find them and fix them.
- The answer to this question is [. . .] Can you find a way to work it out?

Identifying where pupils use and extend their existing knowledge:

- You've used substitution to solve all these simultaneous equations. Can you use elimination?
- You seem to be having difficulty adding some of these fractions and not others. In question 2 you used equivalent fractions; could you use this on question 4?

Encouraging pupils to reflect:

- You used two different methods to solve these problems. What are the advantages and disadvantages of each?
- You have understood [. . .] well. Can you make up your own more difficult problems?

Suggesting pupils discuss their ideas with other pupils:

- You seem to be confusing sine and cosine. Talk to Katie about how to work out the difference.
- Compare your work with Ali and write some advice to another student talking this topic for the first time.

Helping pupils to show their working:
Often teachers ask pupils to 'show your working' or to 'improve your presentation'. Our research suggests that a more effective strategy is to use the notion of an audience for a pupil's work.

- The way in which you are presenting graphs is much clearer. Look back at your last work on graphs in February. What advice would you give on how to draw graphs?
- Your solutions are all correct, but they are a bit brief. Look at the examination marking criteria. Work with Leo to produce model answers that would convince the examiner to award you all the marks.

Advice on learning facts and procedures:
Mathematics inevitably involves the learning of facts and procedures, whether this be in the context of number facts or integration. Learning these facts is generally easier if students can connect what they need to know to what they already know:

> There are a few multiplication facts they/ You are getting wrong. You need to look at a multiplication table. Highlight the facts that you know very well. Finds ways of working out the ones you don't know so well. What facts would help you work out 7 x 8?

Having given feedback, the teacher needs to set aside time for students to read, respond to and act upon feedback. Providing quality class time enables students to discuss the feedback with others, and to ask the teacher (or another pupil) for clarification. This will also provide an opportunity for students to write their own notes on what they need to do next.

5. Self- and peer assessment

Self-assessment has an essential role to play in formative practice. Teachers can create wonderful lessons by facilitating debates on ideas and providing guidance on the next learning steps, but it is only the learner who can do the learning. This is not a simple task as it requires students to have a sufficiently clear picture of the targets in the learning trajectory ahead of them and a means of moving forward to close the learning gap. We have found that peer assessment helps students develop and hone their self-assessment skills. As with other formative assessment techniques, students need training; starting in a small way and evolving their practice gradually is the best way forward. Key to the success of both peer and self-assessment is talking about mathematics, as this provides students with ways of talking about mathematical concepts.

Two stars and a wish:
One way of setting an appropriate tone and balance to peer comments is to suggest that pupils identify two things done well and one area to improve.

Commenting in pairs:
Working with a partner to mark an actual or a teacher-constructed piece of work can give students an opportunity to share ideas.

Checking understanding:
At the beginning of a topic, students could be given a set of statement cards (e.g. multiplication always makes numbers bigger, you cannot divide by 0. . .). Working in groups or pairs, they could sort the cards into ideas into 'agree', 'disagree', 'don't know' piles. Students could revisit the activity at the end of the lesson sequence to assess what they have learned.

Discussing comments:
Giving pupils an opportunity to share and discuss comments – whether by other students or the teacher – can be a useful way of honing students' skills and identifying effective types of comments.

Self-assessment can also be used in 'real-time' rather than just at the end of a learning sequence. Some teachers have given each pupil three paper cups – one green, one yellow and one red. At the beginning of the lesson all students have the cups nested so that the green cup is showing. If pupils feel the teacher is going too fast, they can show the yellow cup, and if they want the teacher to stop, they show the red cup. Initially, pupils are often reluctant to show that they do not understand. However, if, when the teacher sees a red cup, he or she calls on a pupil showing green to give an explanation, pupils are suddenly more willing to show that they do not understand.

6. Putting it into practice

The changes in practice recommended here are not easily made. They require changes in the ways teachers work with students, which may seem risky, and which will certainly be challenging. The work we have done with teachers suggests that the teachers who are most successful are those who change their practice slowly, by focusing on only one or two aspects at a time. As they become skilled with these new ideas, and incorporate them into their natural practice, they can then turn their attention to new ideas. Teachers who try to change many things about their practice at the same time are unlikely to be successful.

The other thing that appears to be crucial if teachers are to develop their practice in fruitful ways is support. We know of a small number of examples where teachers have managed to implement radical changes in their practice on their own, but these are rare. Successful development of one's practice is far more likely if one can draw on the support of one's peers, and two forms of support have, in our experience, been particularly important.

The first is to meet regularly – ideally once a month – with other teachers who are trying to make similar changes in their practice. Many teachers have told us that it was the fact that they were going to have to talk to their colleagues about their experiences in trying these ideas out that forced them to try these ideas in their classroom.

The second is to arrange for a trusted peer to observe one's teaching and provide feedback. The crucial feature of such peer observations is that the agenda for the observation must be set by the teacher being observed. Additionally, where the teacher being observed tells the peer not only what to look for but also what would count for the teacher being observed as evidence of success or failure, there is less chance of the observer introducing his or her own biases and prejudices into the process.

This these elements – practical ways for taking small steps in developing one's practice, and support from one's colleagues – formative assessment can produce substantial and sustained improvements in student achievement and make teaching more enjoyable and professionally rewarding.